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International Journal of Solids and Structures 42 (2005) 2837–2850
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Periodic group crack problems in an infinite plate

Y.Z. Chen *, X.Y. Lin

Division of Engineering Mechanics, Jiangsu University, Xuefu Road 301, Zhenjiang, Jiangsu 212013, China

Received 1 April 2004; received in revised form 29 September 2004
Available online 6 November 2004

Abstract

This paper investigates periodic group crack problems in an infinite plate. The periodic group crack is composed of infinite groups with numbering from $j = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots,$ to $j = \infty$, and the groups are placed periodically. The same loading condition and the same geometry are assumed for cracks in all groups. A singular integral equation is used to solve the problems. The singular integral equation is formulated on cracks of the 0th group (or the central group) with the collection of influences from the infinite groups. The influences of many neighboring groups to the central group are evaluated exactly. Meantime, the influences of many remote groups to the central group can be summed up into one term approximately. The stress intensity factors at crack tips can be evaluated from the solution of the singular integral equation. It is found from some sample problems that the obtained results are very accurate. Finally, several numerical examples are presented and interaction among the group cracks is addressed.

Keywords: Periodic group crack; Interaction of cracks; Numerical solution of singular integral equation

1. Introduction

Many researchers studied the multiple crack problems in an infinite plate (Savruk, 1981; Gross, 1982; Parton and Perlin, 1984; Chen, 1984a,b; Kachanov, 1993). Recently, the previously obtained results were collected in a publication (Chen et al., 2003). In the book, two kinds of the singular integral equations, two kinds of Fredholm integral equation and one kind of hypersingular integral equation were developed to solve the multiple crack problems. Solutions of some sample problems showed that the same results had been obtained by using different kinds of integral equation. That is to say, the multiple crack problems in an infinite plate have been solved very well at present.

* Corresponding author: Tel.: +86 05118780780; fax: +86 05118791739.
E-mail address: chens@ujs.edu.cn (Y.Z. Chen).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.09.046

The periodic crack problem is a particular case of the multiple crack problem. Only some simple cases of the problem have been solved previously (Savruk, 1981). For example, the infinite collinear crack problem was solved in an earlier time (Westergaard, 1939). The problem of infinite rows of parallel cracks was solved by using the integral transform method (Sneddon, 1973). For an infinite plate weakened by periodic cracks, the relevant boundary problem was solved by using the eigenfunction expansion variational method (Chen and Lee, 2002). Recently, elastic analysis for a row of collinear holes in an infinite plate was solved by using the hypersingular integral equation (Wang et al., 2003). In the solution, the influence of holes at the remote place to the central hole was neglected.

For the simple periodic crack problem, a singular integral equation is generally formulated on the central crack. In the problem, the influences caused by many neighboring cracks can be summed up in one term. In this case, the kernel in the integral equation takes a simple form and the relevant numerical solution can be obtained without any difficulty (Savruk, 1981; Chen et al., 2003).

This paper investigates periodic group crack problems in an infinite plate. The periodic group crack is composed of infinite groups with numbering from $j = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots$, to $j = \infty$, and the groups are placed periodically. The same loading condition and the same geometry are assumed for cracks in all groups. Not like the periodic crack problem, in the periodic group crack problems one cannot sum up all influences of the remote groups to the central group into an explicit expression. This is a difficult point encountered. The mentioned difficult point was overcome in the following way. A singular integral equation is used to solve the problem. The singular integral equation is formulated on cracks of the 0th group (or the central group) with the collection of influences from the infinite groups. Here, the influences represent the kernel in the integral equation. The influences of many neighboring groups ($j = -(N-1)$, $-(N-2), \ldots, -2, -1, 1, 2, \ldots, (N-2), (N-1)$) to the central group are evaluated exactly. From the structure of the kernel we found that the influence caused by the $-N$th group and the Nth group to the central group is directly proportional to the factor $1/N^2$. With this property the influences of many remote groups ($j = -\infty, \ldots, -(N+2), -(N+1), -N, N, (N+1), (N+2), \ldots, \infty$) to the central group can be summed up in one term approximately with sufficient accuracy.

The stress intensity factors at crack tips in the problems can be evaluated from the solution of the singular integral equation. It is found from some sample problems that the obtained results are very accurate. In two examples, the relative error for the value of stress intensity factors (abbreviated as SIFs) at crack tips is less than 0.01%. Finally, several numerical examples are presented and interaction among group cracks is addressed.

2. Singular integral equation for multiple crack problems

The fundamentals of the complex variable function method, which plays an important role in plane elasticity, are briefly introduced in what follows. In the method, the stresses ($\sigma_x, \sigma_y, \sigma_{xy}$), the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of the complex potentials $\phi(z)$ and $\psi(z)$ such that (Muskhelishvili, 1953)

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\Phi(z)$$

$$\sigma_y - i\sigma_{xy} = 2\,\mathrm{Re}\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)} \tag{1}$$

$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{2}$$

$$2G(u + \mathrm{i}v) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{3}$$

where $\Phi(z) = \phi'(z)$, $\Psi(z) = \psi'(z)$, a bar over a function denotes the conjugated value for the function, G is the shear modulus of elasticity, $\kappa = (3 - v)/(1 + v)$ in the plane stress problem, $\kappa = 3 - 4v$ in the plane strain problem, and v is the Poisson's ratio.

The loading condition for the multiple crack problem is shown in Fig. 1(a). The stresses applied on the cracks are assumed in the form

$$(\sigma_y - i\sigma_{xy})_k = p_k(s_k) - iq_k(s_k), \quad (|\, s_k \,| < a_k, k = 1, 2, \ldots, N) \tag{4}$$

where $p_k(s_k)$ and $q_k(s_k)$ are the normal and shear stresses which are expressed in the local coordinates $x_k o_k y_k$.

One way to model the multiple crack problem is as follows. The multiple crack problem shown in Fig. 1(a) can be considered as a superposition of N single crack problems, with undetermined distributed dislocations $g'_k(s_k)(|\, S_k \,| < a_k,\ k = 1, 2, \ldots, N)$ on the cracks (Fig. 1(b)). After some manipulations, a system of the singular integral equations can be formulated as follows (Savruk, 1981; Chen et al., 2003):

$$\frac{1}{\pi}\int_{-a_k}^{a_k} \frac{g'_k(t)\mathrm{d}t}{t - s_k} + \frac{1}{\pi}\sum_{j=1}^{N}{}' \int_{-a_j}^{a_j} [g'_j(s_j)K_{jk}(s_j, s_k) + \overline{g'_j(s_j)}L_{jk}(s_j, s_k)]\mathrm{d}s_j = p_k(s_k) - iq_k(s_k)$$
$$(|\, s_k \,| < a_k,\ k = 1, 2, \ldots, N) \tag{5}$$

where the symbol $\sum'$ means that the term corresponding to $j = k$ has been excluded in the summation. The kernels in Eq. (5) may be written as

$$K_{jk}(s_j, s_k) = \frac{1}{2}\exp(\mathrm{i}\alpha_j)\left[\frac{1}{T_j - T_k} + \exp(-2\mathrm{i}\alpha_k)\frac{1}{\overline{T}_j - \overline{T}_k}\right],$$

$$L_{jk}(s_j, s_k) = \frac{1}{2}\exp(-\mathrm{i}\alpha_j)\left[\frac{1}{\overline{T}_j - \overline{T}_k} - \exp(-2\mathrm{i}\alpha_k)\frac{T_j - T_k}{(\overline{T}_j - \overline{T}_k)^2}\right] \tag{6}$$

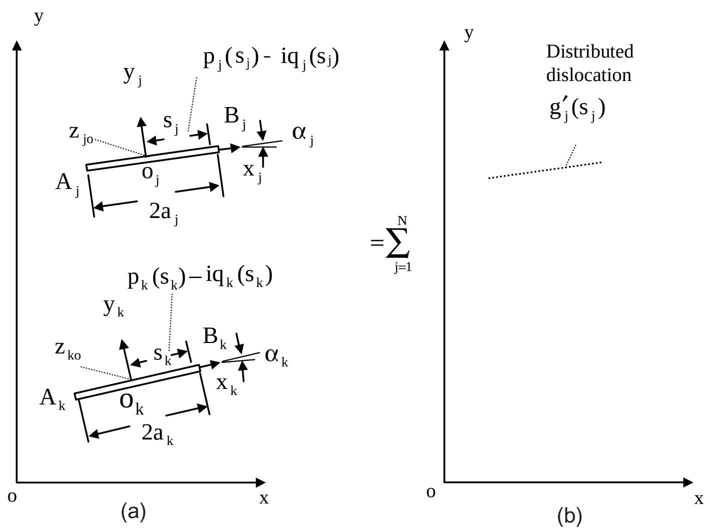

Fig. 1. Superposition method for the multiple crack problem: (a) the original problem and (b) superposition by the distributed dislocations.

where

$$T_k = z_{ko} + s_k \exp(i\alpha_k), \quad (k = 1, 2, \ldots, N) \tag{7}$$

Also, the meaning of a_k, z_{ko}, s_k and α_k $(k = 1, 2, \ldots, N)$ has been indicated in Fig. 1(a). Physically, for example, the kernel $K_{jk}(s_j, s_k)$ and $L_{jk}(s_j, s_k)$ represents the influences of the jth crack to the kth crack.

In Eq. (5) the dislocation distribution functions are defined by

$$g_j'(s_j) = -\frac{2Gi}{\kappa+1}\frac{\mathrm{d}[(u_j(s_j)+iv_j(s_j))^+ - (u_j(s_j)+iv_j(s_j))^-]}{\mathrm{d}s_j} \quad (|\, s_j \,| < a_j,\ j = 1, 2, \ldots, N) \tag{8}$$

where $[(u_j(s_j)+iv_j(s_j))^+ - (u_j(s_j)+iv_j(s_j))^-]$ stands for the jump value of the displacements along the jth crack, and $(u_j(s_j)+iv_j(s_j))^+$, $((u_j(s_j)+iv_j(s_j))^-)$ denotes the displacements at a point "s_j" of the upper (lower) face of the jth crack.

Meantime, the single-valuedness condition of the displacements gives the following constraint equation:

$$\int_{-a_k}^{a_k} g_k'(t)\mathrm{d}t = 0 \quad (k = 1, 2, \ldots, N) \tag{9}$$

It is seen that the mentioned function $g_k'(t)$ has a particular character at the vicinity of the crack tips, and it can be expressed in the form:

$$g_k'(t) = G_k(t)(a_k^2 - t^2)^{-1/2} \quad (|\, t \,| < a_k, \quad k = 1, 2, \ldots, N) \tag{10}$$

After the singular integral equation is solved, the SIFs at crack tip may be defined by

$$(K_1 - iK_2)_{A,k} = (2\pi)^{1/2} \operatorname*{Lim}_{t \to -a_k} \sqrt{|\, t + a_k \,|}\, g_k'(t) = (\pi/a_k)^{1/2} G_k(-a_k)$$

$$(K_1 - iK_2)_{B,k} = -(2\pi)^{1/2} \operatorname*{Lim}_{t \to a_k} \sqrt{|\, t - a_k \,|}\, g_k'(t) = -(\pi/a_k)^{1/2} G_k(a_k) \quad (k = 1, 2, \ldots, N) \tag{11}$$

where the subscript A (or B) is for left (or right) crack tip, respectively, and the subscript "k" means that the equation is used for the kth crack.

For solving the integral equation (5) and Eq. (9) numerically, the following integration rules are useful (Savruk, 1981; Chen et al., 2003):

$$\int_{-a}^{a} \frac{G(t)\mathrm{d}t}{\sqrt{a^2 - t^2}(t - s_m)} = \frac{\pi}{M}\sum_{j=1}^{M} \frac{G(t_j)}{t_j - s_m} \quad (m = 1, 2, \ldots, M-1, M - \text{integer}) \tag{12}$$

$$\int_{-a}^{a} \frac{G(t)\mathrm{d}t}{\sqrt{a^2 - t^2}} = \frac{\pi}{M}\sum_{j=1}^{M} G(t_j) \tag{13}$$

where

$$t_j = a\cos\frac{(2j-1)\pi}{2M} \quad (j = 1, 2, \ldots, M) \tag{14}$$

$$s_m = a\cos\frac{m\pi}{M} \quad (m = 1, 2, \ldots, M-1) \tag{15}$$

In Eqs. (12) and (13), M represents the number of the abscissas in integration. Physically, all integrated functions ($G_k(t)$, $k = 1, 2, \ldots, N$) in the present study are smooth functions. Based on this property, accurate results can be obtained by using $M \geqslant 15$ in Eqs. (12) and (13). This is followed from a computational experience (Chen et al., 2003).

If the $G(t_j)$ $(j = 1, 2, \ldots, M)$ values are known beforehand, the $G(-a)$ and $G(a)$ values can be evaluated by the following extrapolation formulae:

$$G(-a) = \frac{1}{M}\sum_{j=1}^{M}(-1)^{j+M}G(t_j)\tan((2j-1)\pi/4M)$$

$$G(a) = \frac{1}{M}\sum_{j=1}^{M}(-1)^{j+1}G(t_j)\cot((2j-1)\pi/4M) \tag{16}$$

After substituting Eq. (10) into Eqs. (5) and (9), and using the mentioned integration rules, a system of algebraic equations for the values of the function $G_k(t)$ at the discrete points is obtainable. If one choose the same M value for each equation in Eqs. (5) and (9), the unknowns are $G_k(t_j)$ $(j = 1, 2, \ldots, M, k = 1, 2, \ldots, N)$, and the number of unknowns is $M \times N$. In addition, the number of equations is also $M \times N$, where $(M - 1) \times N$ sets are from Eq. (5) and N sets are from Eq. (9). After the algebraic equations are solved, we can obtain the SIFs by using Eqs. (11) and (16).

3. Solution of the periodic group crack problems

The periodic group crack is composed of infinite groups of cracks from $j = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, j = \infty$, and the groups are placed periodically (Fig. 2(a)). It is assumed that, each group is

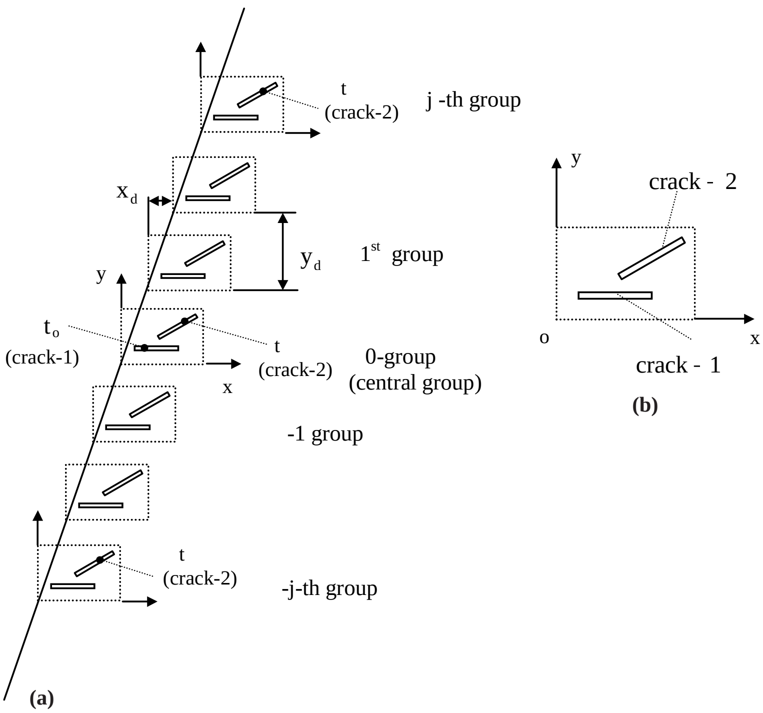

Fig. 2. Periodic group crack problem: (a) periodic group crack and (b) two cracks in a central group.

composed of the same cracks with the same boundary tractions on the crack faces. Without losing generality, in the present study each group is composed of two crack, the crack-1 and the crack-2 (Fig. 2(a)).

The first step is to consider the multiple crack problem for the central group (Fig. 2(b)). Eqs. (5) and (9) are used to obtain the solution for the distribution dislocation function. In the solution, the number of the abscissas (t_j in Eqs. (12) and (14)) is "M", and the number of the collocation points (s_m in Eqs. (12) and (15) is "$M-1$". In this case, after discretization Eq. (5) may be written in the following form:

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_0 \\ \mathbf{D}_0 & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} \tag{17}$$

where $\mathbf{A}_0$ denotes an influence matrix for the crack-1 to itself, $\mathbf{B}_0$ denotes an influence matrix for the crack-2 to the crack-1, $\mathbf{C}_0$ denotes an influence matrix for the crack-2 to itself, and $\mathbf{D}_0$ denotes an influence matrix for the crack-1 to the crack-2. In the present case, all the matrices $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$ have a dimension $(2M-2)\times(2M)$. Meantime, $\mathbf{G}_1$ and $\mathbf{G}_2$ denote a unknown vector with dimension $2M$, which is obtained after discretization for the functions $G_1(t)$ and $G_2(t)$ shown in Eq. (10). Meantime, $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the traction vector with dimension $(2M-2)$, which is obtained after discretization for the functions presented the right hand term of Eq. (5). Note that, the singular integral equation (5) is expressed in a form of complex variable, and computation should be performed in the real value. With this in mind, saying, the matrix $\mathbf{A}_0$ should have a dimension $(2M-2)\times(2M)$, rather than $(M-1)\times(M)$.

Furthermore, the algebraic equation (17) in conjunction with a discretization of Eq. (9) will give the final solution for the dislocation distribution functions. Later, the SIFs at crack tips can be evaluated by using Eqs. (10) and (11). The detail of computation can be found from (Chen et al., 2003).

The periodic group crack problem will be studied below. Physically, if one collects all the influences from the groups $j=-\infty,\ldots,-2,-1,0,1,2,\ldots,j=\infty$ to the central group, an algebraic equation on the cracks for the central group can be formulated. The mentioned equation has the following form:

$$\left[\begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_0 \\ \mathbf{D}_0 & \mathbf{C}_0 \end{bmatrix} + \sum_{j=1}^{N-1}\left[\begin{bmatrix} \mathbf{A}_{-j} & \mathbf{B}_{-j} \\ \mathbf{D}_{-j} & \mathbf{C}_{-j} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_j & \mathbf{B}_j \\ \mathbf{D}_j & \mathbf{C}_j \end{bmatrix}\right] + \sum_{j=N}^{\infty}\left[\begin{bmatrix} \mathbf{A}_{-j} & \mathbf{B}_{-j} \\ \mathbf{D}_{-j} & \mathbf{C}_{-j} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_j & \mathbf{B}_j \\ \mathbf{D}_j & \mathbf{C}_j \end{bmatrix}\right]\right] \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} \tag{18}$$

The matrices, for example $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}_j$ and $\mathbf{D}_j$ ($j=-\infty,\ldots,-2,-1,0,1,2,\ldots,j=\infty$) represent the influence of the jth group to the central group, and they can be evaluated from the kernels in Eq. (5). In more detail, for example, matrix $\mathbf{B}_j$ denotes an influence matrix for the crack-2 of jth group to the crack-1 of the central group, which is similar to the matrix $\mathbf{B}_0$ mentioned above.

It is a key point in the analysis to obtain the summation $\sum_{j=N}^{\infty}$ in Eq. (18) approximately. It is seen from Eq. (5) that the influence at a point "t_o" of the crack-1 of central group caused by a concentrated source at a point "t" of the crack-2 of the central group is proportional to $1/(t-t_o)$. With this estimation, the influence at a point "t_o" of the crack-1 of central group caused by a concentrated source at a point "t" of the crack-2 of the -jth and the jth groups must be proportional to (Fig. 2(a))

$$\frac{1}{t+jz_d-t_o}+\frac{1}{t-jz_d-t_o}=\frac{2(t-t_o)}{(t-t_o)^2-(jz_d)^2}\approx-\frac{2(t-t_o)}{z_d^2}\frac{1}{j^2} \quad \text{(for large } j\text{)} \tag{19}$$

where

$$z_d = x_d + \mathrm{i}y_d$$

and (x_d, y_d) are the periodic distances between two neighboring groups (Fig. 2(a)).

It is expected that each element in the matrix $\mathbf{A}_{-j}+\mathbf{A}_j, \mathbf{B}_{-j}+\mathbf{B}_j, \mathbf{C}_{-j}+\mathbf{C}_j$ and $\mathbf{D}_{-j}+\mathbf{D}_j$ (for large j) has the same property as shown by Eq. (19). From this property, the following approximation is obtained:

$$\frac{1}{j^2}\left[\begin{bmatrix}\mathbf{A}_{-j} & \mathbf{B}_{-j}\\ \mathbf{D}_{-j} & \mathbf{C}_{-j}\end{bmatrix}+\begin{bmatrix}\mathbf{A}_{j} & \mathbf{B}_{j}\\ \mathbf{D}_{j} & \mathbf{C}_{j}\end{bmatrix}\right]\approx\frac{1}{(j+1)^2}\left[\begin{bmatrix}\mathbf{A}_{-(j+1)} & \mathbf{B}_{-(j+1)}\\ \mathbf{D}_{-(j+1)} & \mathbf{C}_{-(j+1)}\end{bmatrix}+\begin{bmatrix}\mathbf{A}_{(j+1)} & \mathbf{B}_{(j+1)}\\ \mathbf{D}_{(j+1)} & \mathbf{C}_{(j+1)}\end{bmatrix}\right]\quad(\text{for large } j) \tag{20}$$

From Eq. (20), one may obtain the following approximation:

$$\sum_{j=N}^{\infty}\left[\begin{bmatrix}\mathbf{A}_{-j} & \mathbf{B}_{-j}\\ \mathbf{D}_{-j} & \mathbf{C}_{-j}\end{bmatrix}+\begin{bmatrix}\mathbf{A}_{j} & \mathbf{B}_{j}\\ \mathbf{D}_{j} & \mathbf{C}_{j}\end{bmatrix}\right]\approx\delta\left[\begin{bmatrix}\mathbf{A}_{-N} & \mathbf{B}_{-N}\\ \mathbf{D}_{-N} & \mathbf{C}_{-N}\end{bmatrix}+\begin{bmatrix}\mathbf{A}_{N} & \mathbf{B}_{N}\\ \mathbf{D}_{N} & \mathbf{C}_{N}\end{bmatrix}\right]\quad(\text{for large } N) \tag{21}$$

where

$$\delta=\left(\sum_{j=N}^{\infty}\frac{1}{j^2}\right)\Big/\left(\frac{1}{N^2}\right)=N^2\left(\frac{\pi^2}{6}-\sum_{j=1}^{N-1}\frac{1}{j^2}\right) \tag{22}$$

We may summarize the process of the solution as follows:

(a) We evaluate the final solution for the dislocation distribution functions from the algebraic equation (18) in conjunction with a discretization of Eq. (9).
(b) The matrices in Eq. (18), $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$ and $\mathbf{D}_0$, $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}_j$ and $\mathbf{D}_j$ ($j=-N,\ldots,-2,-1,1,2,\ldots,N$) are evaluated exactly.
(c) The third term in the left-hand side of Eq. (18) is evaluated approximately by using Eq. (21).
(d) The SIFs at the crack tips can be evaluated from the obtained solution for the dislocation distribution functions.

4. Numerical examples

Some numerical examples are given to illustrate the results of the method presented. The first two examples are devoted to examine the accuracy of the presented method.

4.1. Example 1

In the first example, a problem for the infinite collinear cracks with the same length is considered (Fig. 3(a)). The loading in the problem is the remote tension $\sigma_y^\infty = p$. In computation, $M = 25$ is assumed for the number of the abscissas in Eq. (12). The infinite collinear cracks are assumed in a form of infinite groups marked with a dashed line (Fig. 3(a)). Two conditions of computation are assumed.

(a) One considers the influence of the remainders in Eq. (18) (the third term in Eq. (18) with the suffix $\sum_{j=N}^{\infty}$). Meantime, the term is approximated by using Eq. (21).
(b) One does not consider the influence of the remainders in Eq. (18) (the third term in Eq. (18) with the suffix $\sum_{j=N}^{\infty}$). This means that this term is neglected in computation.

In both cases, $N = 10, 20, \ldots, 100$ are assumed. The calculated results for the SIFs at the crack tips are expressed as

$$K_1 = F_1(N, 2a/d)p\sqrt{\pi a} \tag{23}$$

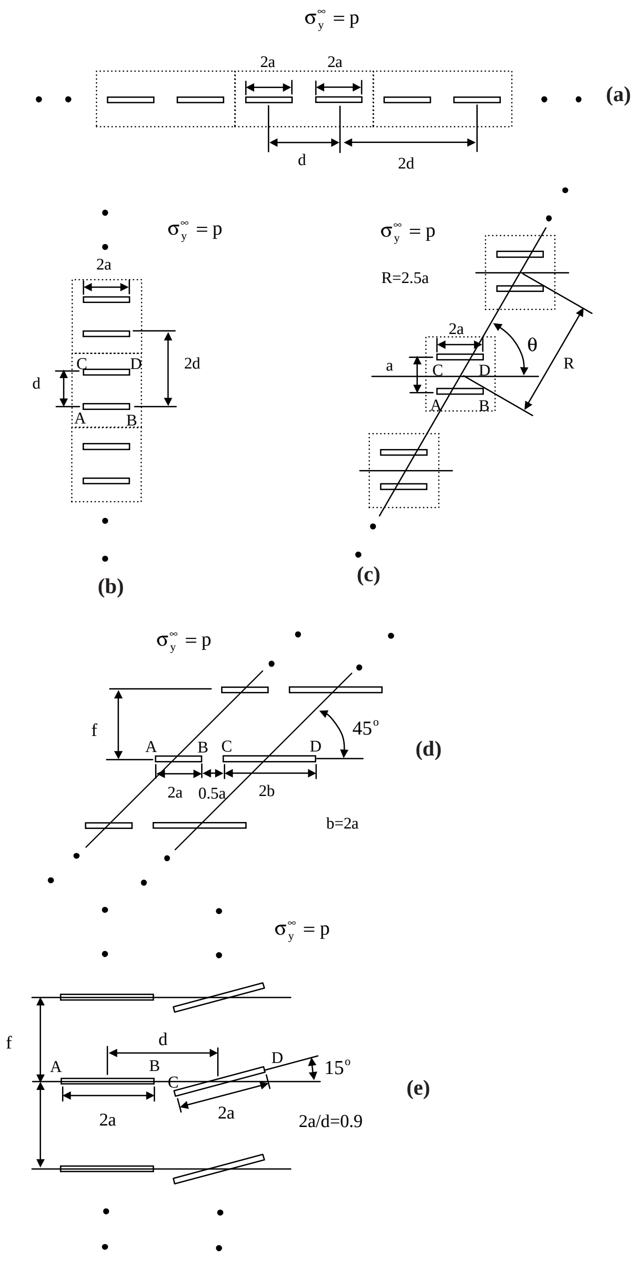

Fig. 3. Five cases of the periodic group crack: (a) infinite collinear cracks; (b) infinite cracks in a stacked position; (c) infinite groups in an inclined position; (d) infinite groups composed of two cracks with unequal length and (e) infinite groups in a stacked position.

The calculated results are plotted in Table 1. In the following estimation, the relative error for the stress intensity factors is denoted by Δ. The $2a/d = 0.9$ case is taken for examination. For $N = 10, 20, \ldots, 100$, we find $|\Delta| < 0.01\%$ under the condition of considering the remainders in Eq. (18).

Table 1
Nondimensional SIFs $F_1(N, 2a/d)$ at crack tips for infinite collinear cracks (see Fig. 3(a) and Eq. (23))

2a/d	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Considering the remainders in Eq. (18), M = 25, N = 10, 20, ..., 100									
N									
10	1.00414	1.01698	1.03983	1.07533	1.12839	1.20848	1.33603	1.56503	2.11342
20	1.00414	1.01698	1.03983	1.07533	1.12838	1.20847	1.33601	1.56498	2.11332
30	1.00414	1.01698	1.03983	1.07533	1.12838	1.20847	1.33601	1.56498	2.11331
40	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
50	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
60	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
70	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
80	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
90	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
100	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
*	1.00415	1.01698	1.03983	1.07533	1.12838	1.20847	1.33601	1.56497	2.11331
Considering the remainders in Eq. (18), N = 50, M = 9, 11, ..., 25									
M									
9	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33599	1.56478	2.10909
11	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33600	1.56495	2.11227
13	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11305
15	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11324
17	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11329
19	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
21	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
23	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
25	1.00414	1.01698	1.03983	1.07533	1.12838	1.20846	1.33601	1.56497	2.11331
*	1.00415	1.01698	1.03983	1.07533	1.12838	1.20847	1.33601	1.56497	2.11331
Not considering the remainders in Eq. (18), M = 25, N = 10, 20, ..., 100									
N									
10	1.00402	1.01649	1.03867	1.07313	1.12462	1.20229	1.32589	1.54742	2.07611
20	1.00408	1.01673	1.03924	1.07420	1.12645	1.20530	1.33081	1.55594	2.09410
30	1.00410	1.01681	1.03943	1.07457	1.12708	1.20633	1.33251	1.55889	2.10036
40	1.00411	1.01685	1.03953	1.07476	1.12740	1.20686	1.33337	1.56039	2.10354
50	1.00412	1.01688	1.03959	1.07487	1.12760	1.20718	1.33389	1.56129	2.10547
60	1.00412	1.01690	1.03963	1.07495	1.12772	1.20739	1.33424	1.56190	2.10676
70	1.00413	1.01691	1.03966	1.07500	1.12782	1.20754	1.33449	1.56234	2.10769
80	1.00413	1.01692	1.03968	1.07504	1.12789	1.20766	1.33468	1.56266	2.10838
90	1.00413	1.01692	1.03970	1.07507	1.12794	1.20775	1.33482	1.56292	2.10893
100	1.00413	1.01693	1.03971	1.07510	1.12799	1.20782	1.33494	1.56312	2.10936

* From an exact solution (Murakami, 1987).

In computation we choose $M = 25$. This is from an experience for doing similar computation (Chen et al., 2003). For the examination of the convergence for the used M, the problem is solved under conditions $M = 9, 11, \ldots, 25$ and $N = 50$. The calculated results are also listed in Table 1. From Table 1 we see that even we take $M = 15$, accurate results are obtained.

Meantime, in the condition of not considering the remainders in Eq. (18), we find $\Delta = -0.91\%$ (for $N = 20$) $\Delta = -0.37\%$ (for $N = 50$), $\Delta = -0.19\%$ (for $N = 100$), respectively. Comparison results prove that the suggested technique provides an effective way for the numerical solution of the periodic group crack problems.

4.2. Example 2

In the second example, a problem for infinite cracks with the same length is in a stacked position (Fig. 3(b)). The loading in the problem is the remote tension $\sigma_y^\infty = p$. In computation, $M = 25$ is assumed for the number of the abscissas in Eq. (12). The infinite cracks are assumed in a form of infinite groups marked with a dashed line (Fig. 3(b)). As before, two conditions of computation mentioned in the Example 1 are also assumed.

In both cases, $N = 10, 20, \ldots, 100$ are assumed. The calculated results for the SIFs at the crack tips are expressed as

$$K_1 = G_1(N, d/a)p\sqrt{\pi a} \tag{24}$$

The calculated results are plotted in Table 2. Similarly, the $d/a = 0.3$ case is taken for examination. For the condition of considering the remainders in Eq. (18), we find $\Delta = 0.18\%$ (for $N = 20$), $\Delta = 0.01\%$ (for $N = 50$) and $\Delta = 0.00\%$ (for $N = 100$), respectively. In this case, the condition $N = 50$ can provide an accurate result. However, for the condition of not considering the remainders in Eq. (18), we find $\Delta = 8.06\%$ (for $N = 20$) $\Delta = 3.16\%$ (for $N = 50$), $\Delta = 1.57\%$ (for $N = 100$), respectively. In this case, even $N = 100$ is taken, one cannot get an accurate result.

Table 2
Nondimensional SIFs $G_1(N,d/a)$ at crack tips for infinite cracks in a stacked position (see Fig. 3(b) and Eq. (24))

d/a	0.3	0.4	0.5	0.6	1.0	2.0	5.0	10.0	20.0
Considering the remainders in Eq. (18)									
N									
10	0.22107	0.25358	0.28280	0.30945	0.39902	0.57019	0.84787	0.95406	0.98789
20	0.21891	0.25250	0.28220	0.30908	0.39895	0.57019	0.84787	0.95406	0.98789
30	0.21864	0.25237	0.28213	0.30904	0.39894	0.57019	0.84788	0.95406	0.98789
40	0.21857	0.25234	0.28211	0.30903	0.39893	0.57019	0.84788	0.95406	0.98789
50	0.21854	0.25233	0.28210	0.30902	0.39893	0.57019	0.84788	0.95406	0.98789
60	0.21853	0.25232	0.28210	0.30902	0.39893	0.57019	0.84788	0.95406	0.98789
70	0.21852	0.25232	0.28210	0.30902	0.39893	0.57019	0.84788	0.95406	0.98789
80	0.21852	0.25232	0.28210	0.30902	0.39893	0.57019	0.84788	0.95406	0.98789
90	0.21851	0.25232	0.28210	0.30902	0.39893	0.57019	0.84788	0.95406	0.98789
100	0.21851	0.25231	0.28210	0.30902	0.39893	0.57019	0.84788	0.95406	0.98789
*	0.21851	0.25231	0.28210	0.30902	0.39893	0.57019	0.84788		
Not considering the remainders in Eq. (18)									
N									
10	0.25433	0.28247	0.30845	0.33261	0.41612	0.58099	0.85198	0.95536	0.98824
20	0.23612	0.26721	0.29518	0.32077	0.40758	0.57567	0.84997	0.95472	0.98807
30	0.23013	0.26218	0.29078	0.31684	0.40470	0.57387	0.84929	0.95451	0.98801
40	0.22717	0.25968	0.28859	0.31487	0.40326	0.57295	0.84894	0.95439	0.98798
50	0.22541	0.25819	0.28728	0.31370	0.40240	0.57241	0.84873	0.95433	0.98796
60	0.22424	0.25720	0.28641	0.31292	0.40182	0.57204	0.84859	0.95428	0.98795
70	0.22341	0.25650	0.28579	0.31236	0.40141	0.57178	0.84848	0.95425	0.98794
80	0.22279	0.25597	0.28533	0.31194	0.40110	0.57158	0.84841	0.95423	0.98793
90	0.22231	0.25556	0.28497	0.31161	0.40086	0.57143	0.84835	0.95421	0.98793
100	0.22193	0.25524	0.28468	0.31135	0.40067	0.57130	0.84830	0.95419	0.98792

* From an exact solution (Murakami, 1987).

4.3. Example 3

In the third example, two cracks with equal length are placed in a group and remote tension $\sigma_y^\infty = p$ is assumed (Fig. 3(c)). The distances between two groups are $x_d = R\cos\theta$, $y_d = R\sin\theta$ (see Figs. 2(a), 3(c)), and $R = 2.5a$. $M = 25$ is assumed for the number of the abscissas in Eq. (12), and $N = 50$ is used in Eqs. (18) and (21). The calculated results for the SIFs at the crack tips are expressed as

$$\begin{aligned} K_{1A} &= K_{1D} = F_{1A}(\theta)p\sqrt{\pi a} \\ K_{1B} &= K_{1C} = F_{1B}(\theta)p\sqrt{\pi a} \\ K_{2A} &= -K_{2D} = F_{2A}(\theta)p\sqrt{\pi a} \\ K_{2B} &= -K_{2C} = F_{2B}(\theta)p\sqrt{\pi a} \end{aligned} \tag{25}$$

For $\theta = 0°, 10°, \ldots, 90°$ par, the calculated results are plotted in Fig. 4. Since all values of SIFs for I-mode are positive, validity of the calculated results is proved.

In the case of $\theta = 0°$, infinite groups are placed in a horizontal position. In this case, the I-mode SIFs at the crack tips possess generally a larger value. For example, $F_{1A}(\theta)|_{\theta=0°} = 1.4959$. The largest value of $F_{1A}(\theta)$ reaches at an intermediate position of θ, and the value is $F_{1A}(\theta)|_{\theta=30°} = 2.1028$. Meantime, in the case of $\theta = 90°$, infinite groups are placed in a stacked position. In the latter case, the I-mode SIFs at the crack tips possess generally a smaller value, for example, $F_{1A}(\theta)|_{\theta=90°} = 0.4463$.

4.4. Example 4

In the fourth example, two cracks with unequal length, one is "2a" and the other is "2b", are placed in a group, and remote tension $\sigma_y^\infty = p$ is assumed (Fig. 3(d)). The distances between two groups are $x_d = f$, $y_d = f$ (see Fig. 2(a), Fig. 3(d)). $M1 = 17$ and $M2 = 25$ are assumed for the number of the abscissas in Eq. (12), for two cracks, respectively. $N = 50$ is used in Eqs. (18) and (21). The calculated results for the SIFs at the crack tips are expressed as

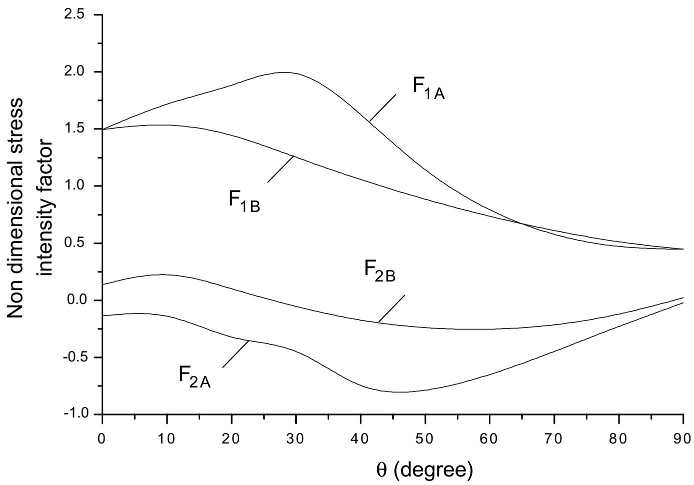

Fig. 4. Nondimensional SIFs $F_{1A}(\theta)$, $F_{1B}(\theta)$, $F_{2A}(\theta)$ and $F_{2B}(\theta)$ (see Fig. 3(c) and Eq. (25)).

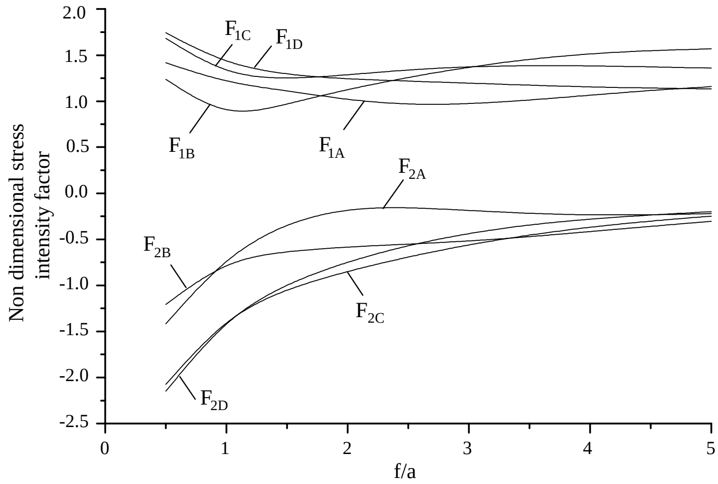

Fig. 5. Nondimensional SIFs $F_{1A}(f/a)$, $F_{1B}(f/a)$, $F_{2A}(f/a)$, $F_{2B}(f/a)$, $F_{1C}(f/a)$, $F_{1D}(f/a)$, $F_{2C}(f/a)$ and $F_{2D}\ (f/a)$ (see Fig. 3(d) and Eq. (26)).

$$
\begin{aligned}
K_{1A} &= F_{2A}(f/a)p\sqrt{\pi a}, \quad K_{1B} = F_{1B}(f/a)p\sqrt{\pi a} \\
K_{2A} &= F_{2A}(f/a)p\sqrt{\pi a}, \quad K_{2B} = F_{2B}(f/a)p\sqrt{\pi a} \\
K_{1C} &= F_{1C}(f/a)p\sqrt{\pi a}, \quad K_{1D} = F_{1D}(f/a)p\sqrt{\pi a} \\
K_{2C} &= F_{2C}(f/a)p\sqrt{\pi a}, \quad K_{2D} = F_{2D}(f/a)p\sqrt{\pi a}
\end{aligned} \tag{26}
$$

For $f/a = 0.5, 1.0, \ldots, 5.0$, the calculated results are plotted in Fig. 5. Since all values of SIFs for I-mode are positive, validity of the calculated results is proved. Form Fig. 5 we see that, in the $f/a = 0.5$ case, or a narrow spacing between groups, the II-mode SIF can reach a considerable value, saying,

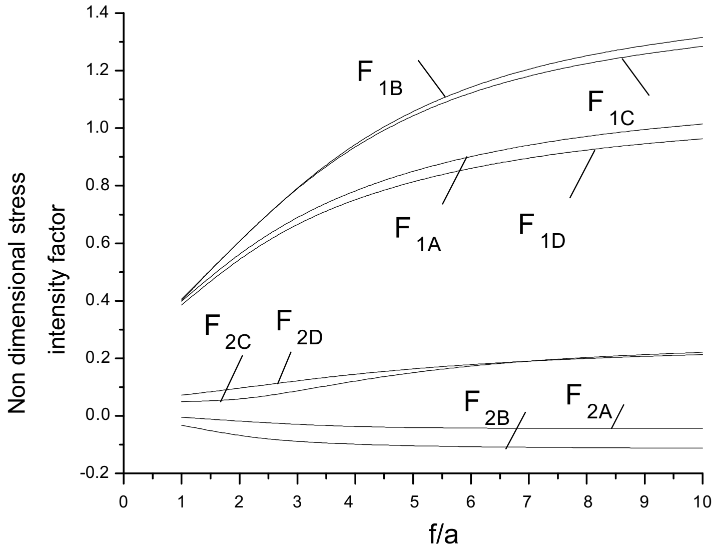

Fig. 6. Nondimensional SIFs $F_{1A}(f/a)$, $F_{1B}(f/a)$, $F_{2A}(f/a)$, $F_{2B}(f/a)$, $F_{1C}(f/a)$, $F_{1D}(f/a)$, $F_{2C}(f/a)$ and $F_{2D}\ (f/a)$ (see Fig. 3(e) and Eq. (26)).

$F_{2D}(f/a)|_{f/a=0.5} = -2.1492$. Because of a particular geometry in the problem, the normal loading can initiate a rather larger II-mode SIF. On the contrary, in the $f/a = 5.0$ case, or a larger spacing between groups, the II-mode SIF becomes a small value, saying, $F_{2D}(f/a)|_{f/a=5.0} = -0.2004$.

4.5. Example 5

In the fifth example, two cracks with equal length are placed in a group (Fig. 3(e)). One crack is in horizontal location and the other is in a slight rotation position. The distances between two groups are $x_d = 0$, $y_d = f$ (see Fig. 2(a), 3(e)). $M = 25$ are assumed for the number of the abscissas in Eq. (12), for two cracks, respectively. $N = 50$ is used in Eqs. (18) and (21). The calculated results for the SIFs at the crack tips can also be expressed by Eq. (26).

For $f/a = 1.0, 2.0, \ldots, 10.0$, the calculated results are plotted in Fig. 6. Since all values of SIFs for I-mode are positive, validity of the calculated results is proved. Since groups are placed in a stacked position, we can easily see the effect of the stack. In a case of the narrow spacing between groups, for example of $f/a = 1.0$, we have $F_{1B}\ (f/a)|_{f/a=1.0} = 0.4058$. However in a case of a larger spacing between groups, for example of $f/a = 10.0$, we have $F_{1B}\ (f/a)|_{f/a=10.0} = 1.3154$.

5. Conclusions

The singular integral equation provides an effective way to solve the multiple crack problem, since the formulation in the problem is based on a rigorous derivation and the relevant integration rule possesses high efficiency. For the periodic group crack problem, an essential step was taken in the present study. The influences from the—jth group and the jth group ($j = N, N+1, \ldots, 1, 2, \ldots, \infty$) to the central group can be summed into one term approximately. This result was shown by Eq. (21). The mentioned approximation can give a very accurate numerical result, which was shown in Examples 1 and 2. On the other hand, if one does not use this method and truncates the successful influence kernels up to $N = 50$ terms, the relative error takes $\Delta = 3.16\%$ in the Example 2. Therefore, we can easily see a particular advantage of the suggested method.

Acknowledgement

The project was supported by National Natural Science Foundation of China.

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